

Criteria for the Application of Double Exponential Transformation

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April 25, 2017

Abstract

The double exponential formula was introduced for calculating definite integrals with singular point oscillation functions and Fourier-integrals. The double exponential transformation is not only useful for numerical computations but it is also used in different methods of Sinc theory. In this paper we use double exponential transformation for calculating particular improper integrals. By improving integral estimates having singular final points. By comparison between double exponential transformations and single exponential transformations it is proved that the error margin of double exponential transformations is smaller. Finally Fourier-integral and double exponential transformations are discussed.

MATHEMATICS SUBJECT CLASSIFICATION: 65D30, 65D32.

Key words: numerical integral, double exponential transformation, single exponential, Fourier integral

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Introduction:

The double exponential transformations (DE) is used for evaluation of integrals of an analytic function has end point singularity. This transformation improved frequently transformation like Fourier transformation. The fact that double exponential transformations error is smaller than single exponential transformation before this was seen as intuitive, but here we have proved it through theoretical analysis. Our numerical results are achieved in the maple software with different numerical results compared with Mori [1]. This article defines the criteria for using double exponential transformations. This technique can not be applied to sharp points. Sinc calculations are very accurate and complex. Regarding the above we have reached a numerical conclusion. In the following sections, we have expanded on the application of double exponential transformations worked in the Fourier integrals and also boundary value conditions. In this regard by incorporating new innovations in the lemma and theorems, we have improved on the above results. In [4] Ooura expended animt-type quadrature formula with the same asymptotic performance as the de-formula. Muhammad, Nurmuhammad and Mori the process by which a numerical solution of integral equations by means of the Sinc collocation method based on the double exponential transformation has attracted considerable attention recently [5]. In a mathematically more rigorous manner, optimality of the double exponential formula-functional analysis approach is established by Sugihara [6]. Koshihara and Sugihara made a full study of A Numerical Solution for the SturmLiouville Type eigenvalue Problems employing the double exponential Transformation [8]. Ooura expended a double exponential formula for the Fourier transformation [10]. Stenger used DE Formula in Sinc approximations [7]. This article is organized as follows:

In section 2 we state and prove a comparison between errors in DE and SE transformation. In section 3 we use DE transformation to compute several integrals and compare the result with that of Mori [1]. In section 4 we demonstrate an application of DE in solving a boundary value problem in section 5 we apply the DE transformation to Fourier integral operators and give a novel proof for the auxiliary theorem 2.

1 Conception of Double Exponential Transformation

Let the following integral be

$$\int_a^b f(x) dx, \quad (1.1)$$

such that the interval (a, b) is infinite or half infinite and the function under integral is analytic on (a, b) and perhaps has singular point in $x = a$ or $x = b$ or both, now consider the change of variables below [9]:

$$x = \phi(t), a = \phi(-\infty), b = \phi(+\infty), \quad (1.2)$$

where ϕ is analytic on $(-\infty, +\infty)$ and

$$I = \int_a^b f(\phi(t)) \phi'(t) dt. \quad (1.3)$$

Hence, after the change of variable the integrand decays double exponential:

$$|f(\phi(t)) \phi'(t)| \approx e^{-ce^{|t|}}, |t| \rightarrow \infty, c > 0. \quad (1.4)$$

By using the trapezoidal formula with mesh size h on (1.3) we have

$$I_h = h \sum_{k=-\infty}^{+\infty} f(\phi(kh)) \phi'(kh). \quad (1.5)$$

The above infinite summation is truncated from $k = -N^-$ to $k = N^+$ in computing (1.5) we get

$$I_h^{(N)} = h \sum_{k=-N^-}^{N^+} f(\phi(kh)) \phi'(kh), N = N^+ + N^- + 1 \quad (1.6)$$

The $N+1$ is due to zero point, for example, $N = 30$ we have $N^- = 14, N^+ = 15$ finally $30=15+14+1$. What gives us the authority to truncate n numbers from the infinite summation? In fact $e^{(-ce^{|t|})}$ quickly approaches zero.

For the integral over $(-1, 1)$.

$$\int_{-1}^1 f(x) dx, \quad (1.7)$$

consider,

$$\phi(t) = \tanh\left(\frac{\pi}{2} \sinh(t)\right), \quad (1.8)$$

by substituting the transformation $x = \phi(t) = \tanh(\pi/2 \sinh(t))$ into (1.5) we obtain

$$\phi(t)' = \tanh'(\pi/2 \sinh(t)) \pi/2 \cosh(t) = \left(\frac{1}{\cosh^2(\pi/2 \sinh(t))} \right) \pi/2 \cosh(t). \quad (1.9)$$

Consequently, by using the double exponential formula we will have:

$$I_h^{(N)} = h \sum_{k=-N^-}^{N^+} f(\tanh(\pi/2 \sinh(kh))) \frac{\pi/2 \cosh(kh)}{\cosh^2(\frac{\pi \sinh(kh)}{2})}. \quad (1.10)$$

2 Numerical Experience and Comparison between D.E Approximation Error and S.E Transformation Approximation Error in Sinc Theory

In this section we prove the main results of our paper. We compare two error estimates for the second-orders two point boundary problem exponential due to Horiuchi and Sugihara [2], which combine the double exponential transformation with the Sinc-Galerkin method, and prove the superiority of one estimate over the other.

As we laid out in [1], one considers

$$\tilde{y}''(x) + \tilde{\mu}(x)\tilde{y}'(x) + \tilde{\nu}(x)\tilde{y}(x) = \tilde{\sigma}(x) \quad a < x < b, \quad (2.1)$$

$$\tilde{y}(a) = \tilde{y}(b) = 0,$$

by using the variable transformation

$$x = \varphi(t) \quad a = \varphi(-\infty) \quad b = \varphi(+\infty), \quad (2.2)$$

since, together with the change of notation we have

$$y(t) = \tilde{y}(\varphi(t)), \quad (2.3)$$

transforms to

$$y''(x) + \mu(x)y'(x) + \nu(x)y(x) = \sigma(x) \quad -\infty < t < +\infty, y(-\infty) = y(+\infty) = 0, \quad (2.4)$$

the Sinc-Galerkin method have ability to approximate the solution of (2.4) by combination of Sinc function:

$$y_N(t) = \sum_{k=-n}^n w_k S(k, h)(t), N = 2n + 1. \quad (2.5)$$

Where

$$S(k, h)(t) = \frac{\sin(\pi)/h(x - kh)}{\pi/h(x - kh)}, k = 0, \pm 1, \pm 2, \dots \quad (2.6)$$

based on numerical result and theoretical proof the approximation error estimate by

$$|y(t) - y_N(t)| \leq c' N^{\frac{5}{2}} \exp(-c\sqrt{N}), \quad (2.7)$$

when the true solution $y(t)$ of the transform problem decays single exponentially leads to

$$|y(t)| \leq \alpha \exp(-\beta|t|). \quad (2.8)$$

Furthermore, the approximation error estimate by

$$|y(t) - y_N(t)| \leq c' N^2 \exp\left(\frac{-cN}{\log N}\right), \quad (2.9)$$

when the true solution $y(t)$ of the transform problem decays double exponentially leads to

$$|y(t)| \leq \alpha \exp(-\beta \exp(\gamma|t|)). \quad (2.10)$$

Theorem 2.1. *In The above situation we prove that the approximation error arising from double exponential decay is less than the approximation error arising from single exponential decay.*

Proof. The single exponential approximation error can be estimated by:

$$|y(t) - y_N(t)| \leq c' N^{\frac{5}{2}} \exp(-c\sqrt{N}). \quad (2.11)$$

$$y_N(t) = \sum_{k=-n}^n w_k S(k, h)(t), N = 2n + 1. \quad (2.12)$$

$S(k, h)(t)$ is Sinc function.

The downside is established in accordance with the following decay single exponential $y(t)$,

$$|y(t)| \leq \alpha \exp(-\beta|t|). \quad (2.13)$$

The double exponential approximation error can be estimate by

$$|y(t) - y_N(t)| \leq c' N^2 \exp\left(\frac{-cN}{\log N}\right), \quad (2.14)$$

if the true solution of $y(t)$ the transformed problem decay double exponentially like[1]

$$|y(t)| \leq \alpha \exp(-\beta \exp(\gamma|t|)), \quad (2.15)$$

we will prove that the error in the D.E transformation is much less than the error in the S.E transformation.

For this purpose we need lemma 2.2.

Lemma 2.2. *There exists N_0 such that for all $N > N_0$, the inequality $c'N^2e^{\frac{-cN}{\log N}} < c''N^{\frac{5}{2}}e^{-c\sqrt{N}}$ hold true.*

Proof.

$$\frac{c'}{c''} < N^{\frac{1}{2}}e^{\frac{cN}{\log N}-c\sqrt{N}}. \quad (2.16)$$

It is necessary to be prove:

$$1 < e^{\frac{cN}{\log N}-c\sqrt{N}}, \frac{c'}{c''} < N^{\frac{1}{2}}, \quad (2.17)$$

$$1 < e^{\frac{cN}{\log N}-c\sqrt{N}} \iff 0 < \frac{cN}{\log N} - c\sqrt{N}, \quad (2.18)$$

$$\frac{c'}{c''} < N^{\frac{1}{2}} \iff \left(\frac{c'}{c''}\right)^2 < N, \quad (2.19)$$

$$0 < \frac{cN}{\log N} - c\sqrt{N} \implies cN > c\sqrt{N} \log N \iff \sqrt{N} > \log N \stackrel{N=\log N}{\iff} e^{\frac{\log N}{2}} > \log N, \quad (2.20)$$

according to lemma 2.3 there exists x_0 such that for,

$$x > x_0, e^{\frac{x}{2}} > x, \quad (2.21)$$

is now

$$N > e^{x_0} \implies \log N > x_0 \stackrel{N=\log N}{\iff} e^{\frac{\log N}{2}} > \log N, \quad (2.22)$$

then put $N_0 = \max\{(\frac{c'}{c''})^2, e^{x_0}\}$ and the theorem satisfied. \square

For complete the proof we need other lemma.

Lemma 2.3. *For each $a > 0$ such that $t > 2a$ we have $e^t > at$.*

Proof.

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots > 1 + t + \frac{t^2}{2}, \quad t > 0, \quad (2.23)$$

for this we prove that,

$$1 + t + \frac{t^2}{2} > at, \quad (2.24)$$

this yields,

$$1 + (1 - a)t + \frac{t^2}{2} > 0, \quad (2.25)$$

$$t > \frac{a - 1 + \sqrt{(a - 1)^2 - 2}}{1}, \quad (2.26)$$

and

$$t > \frac{a - 1 - \sqrt{(a - 1)^2 - 2}}{1}, \quad (2.27)$$

the discriminant of the quadratic equation

$$1 + (1 - a)t + \frac{t^2}{2} = 0, \quad (2.28)$$

is

$$\Delta = (1 - a)^2 - 2 = a^2 - 2a - 1 = 0, \quad (2.29)$$

if $\Delta \geq 0$ we put t_0 equal max root in quadratic equation $t_0 = a - 1 + \sqrt{(a - 1)^2 - 2}$. Then the statement is satisfied. And note that

$$t_0 < a + \sqrt{a^2} = 2a. \quad (2.30)$$

If $\Delta < 0$ that is the roots are complex and the coefficient of t^2 is positive then for all of t inequality is true. Hence we can set $t_0 = 0$ and this completes the proof.

□

□

3 NUMERICAL EXPERIENCE

For the following four integral we get:

DEFINT(Table 1)			
INTEGRAL	N	abs. error	
I_1	25	0.370582258	
I_2	387	0.154551243	
I_3	387	0.021640875	
I_4	259	0.001118879	

In the above table, the absolute error tolerance is 10^{-8} where N is the number of function evaluations and abs. Error is the actual absolute error of the result and it gives an approximate value which is correct up to 16 significant digits.

$$I_1 = \int_0^1 X^{\frac{-1}{4}} \log \frac{1}{X} dx. \quad I_2 = \int_0^1 \frac{1}{16(X - \frac{\pi}{4})^2 + \frac{1}{16}} dx. \quad I_3 = \int_0^\pi \cos(64 \sin X) dx.$$

$$I_4 = \int_0^1 \exp(20(x-1)) \sin(256x) dx.$$

Our results are very different from [1] which was Previously defined. Compare quadrature or Horner method, give at least 10^{-5} errors so these method are better than the double exponential transformation method.

4 Application of the Double-Exponential Transformation for the Sinc-Galerkin Method and the Second-Order Two-Point Boundary Problem

Let

$$\tilde{y}''(x) + \tilde{\mu}(x)\tilde{y}'(x) + \tilde{\nu}(x)\tilde{y}(x) = \tilde{\sigma}(x) \quad a < x < b, \quad (4.1)$$

$$\tilde{y}(a) = \tilde{y}(b) = 0,$$

by using the variable transformation

$$x = \varphi(t) \quad a = \varphi(-\infty) \quad b = \varphi(+\infty), \quad (4.2)$$

since, together with the change of notation we have

$$y(t) = \tilde{y}(\varphi(t)), \quad (4.3)$$

and

$$y'(t) = \varphi'(t)\tilde{y}'(\varphi(t)), \quad (4.4)$$

hence

$$y''(t) = \varphi''(t)\tilde{y}'(\varphi(t)) + \varphi'(t)^2\tilde{y}''(\varphi(t)), \quad (4.5)$$

in virtual point that 4.4 we obtain:

$$\tilde{y}'(\varphi(t)) = \frac{1}{\varphi'(t)}y'(t). \quad (4.6)$$

In note that 4.5 we have:

$$\tilde{y}''(x) = -\tilde{\mu}(x)\tilde{y}'(t) - \tilde{\nu}(x)\tilde{y}(x) + \tilde{\sigma}(x), \quad (4.7)$$

$$y''(t) = \varphi''(t)\tilde{y}'(\varphi(t)) + \varphi'(t)^2(-\tilde{\mu}(\varphi(t))\tilde{y}'(\varphi(t)) - \tilde{\nu}(\varphi(t))\tilde{y}(\varphi(t)) + \tilde{\sigma}(\varphi(t))). \quad (4.8)$$

Our aim is change \tilde{y} to y by using 4.3, 4.8 we get follow equation:

$$y''(t) = \varphi''(t)\frac{1}{\varphi'(t)}y'(t) + \varphi'(t)^2(-\tilde{\mu}(\varphi(t))\frac{1}{\varphi'(t)}y'(t) - \tilde{\nu}(\varphi(t))y(t) + \tilde{\sigma}(\varphi(t))), \quad (4.9)$$

this however, leads to

$$y''(t) + (\varphi'(t)\tilde{\mu}(\varphi(t)) - \frac{\varphi''(t)}{\varphi'(t)})y'(t) + \varphi'(t)^2\tilde{\nu}(\varphi(t))y(t) = \varphi'(t)^2\tilde{\sigma}(\varphi(t)), \quad (4.10)$$

$$\mu(t) = (\varphi'(t)\tilde{\mu}(\varphi(t)) - \frac{\varphi''(t)}{\varphi'(t)}), \quad (4.11)$$

$$\nu(t) = \varphi'(t)^2\tilde{\nu}(\varphi(t)), \quad (4.12)$$

$$\sigma(t) = \varphi'(t)^2\tilde{\sigma}(\varphi(t)), \quad (4.13)$$

and observ that

$$y''(t) + \mu(t)y'(t) + \nu(t)y(t) = \sigma(t). \quad (4.14)$$

By using the boundary conditions and 4.4 we have:

$$\begin{cases} \tilde{y}(a) = \tilde{y}(\varphi(-\infty)) = y(-\infty) = 0 \\ \tilde{y}(b) = \tilde{y}(\varphi(\infty)) = y(\infty) = 0 \end{cases}, \quad (4.15)$$

The Sink Galerkin approximation method provides solutions to transform such as the one in 4.14 by combining it linearly with the Sinc function we have:

$$y_N(t) = \sum_{k=-n}^n W_k S(k, h)(t), \quad N = 2n + 1. \quad (4.16)$$

Every method for trapezoidal numerical integration suggest one method in Sinc approximation. Horiuchi and Sugihara combine the Sinc-Galerkin method with the double exponential transformation for the second-order two-point boundary problem [2]. Mori and Nurmuhammad and Muhammad studied the performance of de-sinc method when used in the second order singularly perturbed boundary value problems [3].

5 Evaluation of Fourier-type integrals

The double exponential is useful for several integrals for example Fourier integral but it does not suitable for Fourier-type integrals slowly oscillatory function. Fourier integral can compute from follow formula:

$$\hat{f}(w) = \int_{-\infty}^{+\infty} f(x)e^{ixw}dx. \quad (5.1)$$

$$e^{ixw} = \cos wx - i \sin wx. \quad (5.2)$$

Therefore we can show sin section with I_s sign and cos section with I_c sign then:

$$\begin{cases} I_s = \int_0^\infty f_1(x) \sin wx \\ I_c = \int_0^\infty f_1(x) \cos wx \end{cases}. \quad (5.3)$$

we choose $\varphi(t)$ function applies to the following conditions.

$$\varphi(-\infty) = 0, \varphi(\infty) = \infty, \quad (5.4)$$

if $t \rightarrow -\infty$ double exponentially

$$\varphi'(t) \rightarrow 0, \quad (5.5)$$

if $t \rightarrow \infty$ double exponentially

$$\varphi(t) \rightarrow t, \quad (5.6)$$

if $t \rightarrow 0$ then

$$\exists D|\varphi'(t)| \leq D \exp(-c \exp|t|)|t| > 0 \iff \varphi'(t) \approx \exp(-c \exp|t|) \rightarrow 0 \quad (5.7)$$

by using 5.6 we have:

$$\left| \frac{\varphi(t)}{t} \right| \approx \exp(-c \exp|t|), \quad (5.8)$$

this means having the same exponentially growth ,on the other hand,

$$\varphi(t) - t \approx \exp(-c \exp|t|). \quad (5.9)$$

Now by using change follow variable in transformation I_S, I_C we have:

$$\begin{cases} I_S : X = M\varphi(t)/w \\ I_C : X = M\varphi(t - \frac{\pi}{2M})/w \end{cases} \quad (M = \text{constant}), \quad (5.10)$$

$$X = M\varphi(t)/w \Rightarrow dx = M\varphi'(t)/w dt. \quad (5.11)$$

By using 5.3 we obtain:

$$I_S = \int_0^\infty f_1(x) \sin wx = \int_0^\infty f_1(M\frac{\varphi(t)}{w}) \sin(M\varphi(t)) \frac{M\varphi'(t)}{w} dt. \quad (5.12)$$

Theorem 5.1. Consider

$$\varphi(t) = \frac{t}{1 - \exp(-k \sinh t)}, \quad (5.13)$$

if $t \rightarrow -\infty$ double exponentially $\varphi'(t) \rightarrow 0$.

Proof.

$$\varphi'(t) = \frac{(1 - e^{-k \sinh t}) - k \cosh t e^{-k \sinh t} t}{(1 - e^{-k \sinh t})^2}, \quad (5.14)$$

if $t \rightarrow -\infty$ then

$$\sinh t = \frac{e^{+t} - e^{-t}}{2} \rightarrow -\infty, \quad (5.15)$$

and observe that

$$\varphi'(t) = \frac{1}{1 - e^{-k \sinh t}} - \frac{k \cosh t e^{-k \sinh t}}{(1 - e^{-k \sinh t})^2}, \quad (5.16)$$

if $t \rightarrow -\infty$ then

$$\frac{1}{1 - e^{-k \sinh t}} \rightarrow 0, \quad (5.17)$$

since by 5.16 have

$$= -\frac{k \cosh t e^{-k \sinh t}}{(1 - e^{-k \sinh t})^2}. \quad (5.18)$$

However, if the second fraction of 5.16 is zero. Condition 5.5 is satisfied.

Lemma 5.2. If $t \rightarrow -\infty$ term $\frac{k \cosh t e^{-k \sinh t}}{(1 - e^{-k \sinh t})^2}$ double exponentially approach to zero.

Proof. Consider

$$A = \frac{ktcosh t}{1 - e^{-ksinh t}} \times \frac{e^{-ksinh t}}{1 - e^{-ksinh t}}, \quad (5.19)$$

$$|A| = |A_1| \times |A_2|, \quad (5.20)$$

such that

$$A_1 = \frac{ktcosh t}{1 - e^{-ksinh t}}, \quad (5.21)$$

$$A_2 = \frac{e^{-ksinh t}}{1 - e^{-ksinh t}}, \quad (5.22)$$

and

$$\lim_{t \rightarrow -\infty} A_2 = \lim_{x \rightarrow \infty} \frac{x}{1 - x} = \frac{\infty}{-\infty} = -1, \quad (5.23)$$

so that

$$\lim_{t \rightarrow \infty} |A_2| = 1. \quad (5.24)$$

For t_0 large enough, $|A_2| < 2$,

numerator A_1 term $ktcosh t$ exponentially approach to ∞ but denominator double exponentially approach to ∞ then antecedent disable for neutralize denominator therefore term obtain such that double exponentially approach to zero with c less than k . \square

Claim: For $c = \frac{k}{4}$ we must prove

$$|A_1| < De^{Ce^{|t|}}. \quad (5.25)$$

$sinh t$ has constant 2 in its denominator hence everything less than the value $\frac{k}{2}$ will work well but for the value k itself, the term equal to ∞ .

Lemma 5.3. *There is t_0 such that for $t > t_0$, $|1 - e^{-ksinh t}| > \frac{1}{2}e^{-ksinh t}$.*

Proof.

$$\lim_{t \rightarrow -\infty} \frac{|1 - e^{-ksinh t}|}{e^{-ksinh t}} = \lim_{x \rightarrow \infty} \frac{|1 - x|}{x} = 1, \quad (5.26)$$

since $1 > \frac{1}{2}$ therefore for $t > t_0$ we have:

$$\frac{|1 - e^{-ksinh t}|}{e^{-ksinh t}} > \frac{1}{2}, \quad (5.27)$$

and

$$|A_1| < \frac{ktcosh t}{\frac{1}{2}e^{-ksinh t}}. \quad (5.28)$$

It is trivial instead denominator A_1 in 5.3 from lemma 5.2 We must prove

$$|t| > t_0, |A_1| < D e^{-\frac{k}{4}e^{|t|}}, \quad (5.29)$$

for this aim we must prove

$$|2kt \cosh t e^{k \sinh t} e^{\frac{k}{4}e^{|t|}}| < D, \quad (5.30)$$

as a matter of fact we change denominator A_1 5.28 to numerator then implies 5.29 we get:

$$t \longrightarrow -\infty, |t| = -t, \text{left side} = |2kt \cosh t e^{\frac{k}{2}e^t} e^{\frac{-k}{2}e^{-t}} e^{\frac{k}{4}e^{|t|}}| < D, \quad (5.31)$$

this yields

$$= |2kt \cosh t e^{\frac{k}{2}e^t} e^{\frac{k}{4}e^{-t}}| < D, \quad (5.32)$$

in view of 5.32 we infer that

$$= |2kt \cosh t e^{\frac{k}{2}e^t} e^{\frac{-k}{4}e^{-t}}| < D, \quad (5.33)$$

such that

$$= \lim_{t \longrightarrow -\infty} 2kt \cosh t e^{\frac{k}{2}e^t} e^{\frac{-k}{4}e^{-t}}, \quad (5.34)$$

indeed, we have

$$\lim_{t \longrightarrow -\infty} e^{\frac{k}{2}e^t} = 1, \quad (5.35)$$

and notice that we can write the equation

$$= 2k \lim_{t \longrightarrow -\infty} \cosh t e^{\frac{-k}{4}e^{-t}}, \quad (5.36)$$

instead $\cosh t = \frac{e^t + e^{-t}}{2}$ in 5.36:

$$= 2k \lim_{t \longrightarrow -\infty} \frac{1}{2} (e^t e^{\frac{-k}{4}e^{-t}} + e^{-t} e^{\frac{k}{4}e^{-t}}), \quad (5.37)$$

we infer that

$$= k \lim_{t \longrightarrow -\infty} (e^t e^{\frac{k}{4}e^{-t}} + e^{-t} e^{\frac{-k}{4}e^{-t}}) = 0. \quad (5.38)$$

□

Lemma 5.4.

$$\lim_{t \longrightarrow -\infty} t - \frac{k}{4}e^{-t} = -\infty, \quad (5.39)$$

$$\lim_{t \longrightarrow -\infty} -t - \frac{k}{4}e^{-t} = -\infty. \quad (5.40)$$

Proof.

$$\lim_{t \rightarrow -\infty} t \times \frac{t - \frac{k}{4}e^{-t}}{t} = -\infty, \quad (5.41)$$

for using L'Hôpital's rule we multiply t in denominator and numerator in fact we create $\frac{\infty}{\infty}$.

$$\lim_{t \rightarrow -\infty} t = -\infty, \quad (5.42)$$

and observe that

$$\lim_{t \rightarrow -\infty} \frac{t - \frac{k}{4}e^{-t}}{t} = \lim_{t \rightarrow -\infty} \frac{1 + \frac{k}{4}e^{-t}}{1} = \infty, \quad (5.43)$$

we obtain

$$\lim_{t \rightarrow -\infty} t \times \frac{-t - \frac{k}{4}e^{-t}}{t} = -\infty. \quad (5.44)$$

Hence 5.38 approach to zero therefore 5.31, 5.32 for $t > t_0$ is true.

We proved 5.18 approach to zero then we will prove 5.17 approach to zero.

According the 5.2 we have:

$$\left| \frac{1}{1 - e^{-ksinh t}} \right| < \frac{1}{\frac{1}{2}e^{-ksinh t}}, \quad (5.45)$$

and

$$= \frac{2}{e^{-k(\frac{e^t - e^{-t}}{2})}}, \quad (5.46)$$

so that

$$= \frac{2}{e^{-k\frac{e^t}{2}} e^{k\frac{e^{-t}}{2}}} = \frac{2e^{k\frac{e^t}{2}}}{e^{k\frac{e^{-t}}{2}}}, \quad (5.47)$$

this , however, leads to

$$\lim_{t \rightarrow -\infty} 2e^{k\frac{e^t}{2}} = 1 \implies |t| > t_0, 2e^{k\frac{e^t}{2}} < 1, \quad (5.48)$$

since $t < 0$, absolute t equal $-t$ gives :

$$\left| \frac{1}{1 - e^{-ksinh t}} \right| < \frac{2}{e^{k\frac{e^{-t}}{2}}} = 2e^{-\frac{k}{2}e^{-t}} < De^{-Ce^{|t|}}. \quad (5.49)$$

And the proof is finished, hence the statement is satisfied. □

□

Conclusion. In this paper the concept of the double exponential transformation was used to establish this method in spite of usefulness and being attractive it is not generally, we cannot use this method in all conditions especially in sharp point case and convergence rate slowly. The result in the oscillatory case is shown much better in I_4 than in the other cases I_1, I_2, I_3 (Table 1).

Appendix A: GALERKIN METHOD The Galerkin method, which approximates a solution of an equation of the form

$$(1 - \lambda k)f = g. \quad (5.50)$$

defined on a Banach space X , is simple to describe. We start with a set of basis functions $\{\psi_k\}_{k=1}^n$, and we assume that for each $f \in X$, we can determine a unique set of numbers c_1, \dots, c_n in \mathbb{C} , such that the projection $P_{n:X \rightarrow X}$ is defined, with

$$P_n f = \sum_{k=1}^n c_k \psi_k. \quad (5.51)$$

It means $p_n^2 = p_n$. The Galerkin method then enables us to replace equation 5.50 by the approximate one, namely 5.52,

$$(1 - \lambda p_n k)f_n = p_n g, \quad (5.52)$$

why did we instead? Because system 5.50 is infinite system but new system is system of n equations in n indeterminate.

$$(1 - \lambda p_n k) \sum_{k=1}^n c_k \psi_k = p_n g, \quad (5.53)$$

and observe that

$$p_n g = \sum_{k=1}^n d_k \psi_k, \quad (5.54)$$

such that

$$(1 - \lambda p_n k) \sum_{k=1}^n c_k \psi_k = \sum_{k=1}^n d_k \psi_k, \quad (5.55)$$

this, however, leads to

$$\sum_{k=1}^n c_k \psi_k - \lambda \sum_{k=1}^n p_n k \psi_k = \sum_{k=1}^n d_k \psi_k, \quad (5.56)$$

we write $p_n k \psi_k$ in terms of ψ_j ,

$$p_n k \psi_k = \sum_{j=1}^n c_{kj} \psi_j, \quad (5.57)$$

we infer that

$$\sum_{k=1}^n c_k \psi_k - \lambda \sum_{k=1}^n \sum_{j=1}^n c_{kj} \psi_j = \sum_{k=1}^n d_k \psi_k, \quad (5.58)$$

we will be equal the coefficients ψ_i on both sides,

$$c_i - \lambda \sum_{k=1}^n c_{ki} = d_i, \quad i = 1, \dots, n \quad (5.59)$$

this process produces n equations, n indeterminate.

In practice, we try to select projection as p_n , to be comfortable working with them and $\|k - p_n k\|$ being so small [7].

References

- [1] M. Mori , M. Sugihara, *The Double Exponential Transformation in Numerical Analysis* , Journal Of Computational And Applied Mathematics Vol.127, (2001), 287-299.
- [2] K. Horiouchi, M.Sugihara, *Sinc - Galerkin Method with The Double Exponential Transformation for the Two Point Boundary Problems* , Technical Report, Dept. of Mathematical Engineering, University of Tokyo , (1998), 99-105.
- [3] M.Mori, A.Nurmuhammad, M.Muhammad, *De-Sinc Method for Second Order Singularly Perturbed Boundary Value Problems* , japan j.Indust.Appl.Math, Vol. 26 , (2009), 41-63.
- [4] T. Ooura, *Animt-Type Quadrature Formula with the Same Asymptotic Performance as the De-Formula* , Journal of Computational and Applied Mathematics, Vol. 213 , (2008), 232-239
- [5] M. Muhammad, A. Nurmuhammad, M. Mori, *Numerical Solution of Integral Equations by Means of the Sinc collocation Method Based on the Double Exponential Transformation* , Journal of Computational and Applied Mathematics , Vol. 177 , (2005), 269-286.
- [6] M. Sugihara, *Optimality of The Double Exponential Formula Functional Analysis Approach* , Numer Math, Vol. 75 , (1997).

- [7] F. Stenger, *Numerical Methods Based on Sinc and Analytic Functions* , (1993).
- [8] T. Koshihara, M. Sugihara, *A Numerical Solution for The Sturm-Liouville Type Eigenvalue Problems Employing The Double Exponential Transformation* , Proceedings Of The 1996 Annual Meeting of The Japan Society for Industrial and Applied Mathematics, 1996, pp. 136-137(In Japanese).
- [9] O. Khatibi, A. Khatibi, *An Upper Bound Estimate and Stability for the Global Error of Numerical Integration Using Double Exponential Transformation* , arXive, Preprints.
- [10] T. Ooura, *A Double Exponential Formula for Fourier Transform* , RIMS, Kyoto Univ., Vol. 41 , (2005), 971-978.